Branching Processes for Multiple Curve Modeling

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12th Financial Risks International Forum
Paris, 18-19 March 2019

Research supported by the Europlace Institute of Finance
Post-crisis interest rate markets: an overview

- A universe of rates: OIS, Eonia, Libor/Euribor for different tenors...
  ...and new reference rates will be soon introduced.

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- Pre-crisis market environment ("textbook situation"): different rates are related by simple no-arbitrage relations and compounding rules.
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- Since the 2007-2008 crisis, the credit and funding/liquidity risks implicit in interbank transactions have deeply affected fixed income markets:
  - interbank risk is typically increasing in the tenor (length of the loan).
  - Filipović & Trolle (2013), Gallitschke et al. (2017).
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- Consequences for financial modeling:
  - interbank (Ibor) rates are risky;
  - classical no-arbitrage relations do not hold.
  - persistently low/negative interest rates.

Multiple interest rate curves, where each interest rate (yield) curve is constructed from products depending on a specific tenor (1W, 1M, 3M, 6M, 1Y).

⇒ this is reflected by the presence of spreads between Ibor and OIS rates.
Additive Euribor–Eonia OIS spreads for different tenors

Source: ECB.

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Multiple curve modeling

**Empirical features of spreads**

- generally positive;
- longer tenors are associated to higher spreads;
- strong comovements and common upward jumps;
- volatility clustering.

The modeling approach

Continuous-state branching processes with immigration (CBI processes) to model OIS short rate; multiplicative spreads between Ibor rates and OIS rates.

Main properties

Consistent with empirical features; order relations between Ibor rates associated to different tenors; analytical tractability and efficient valuation formulae (calibration); automatic fit to the initially observed term structures.
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An (incomplete) overview of modelling approaches

- **Fundamental approaches:**

- **Short rate models:**

- **Libor market models and forward models:**

- **HJM models:**

- **Rational models:**

**Textbooks:** Bianchetti and Morini (2013), Grbac and Runggaldier (2015).

**Precursory works:** Jarrow and Turnbull (1996), Douady and Jeanblanc (2002).
Ibor and OIS rates

- $L(t, t, \delta)$: Ibor rate at date $t$ for the period $[t, t + \delta]$; we consider a finite set $\mathcal{D}$ of tenors $\delta_1 < \ldots < \delta_m$, for $m \in \mathbb{N}$. 

\[ \text{From OIS rates we can compute the term structure of OIS zero-coupon bond prices:} \]
\[ B(t, T) = \frac{1}{\mathcal{L}(t, t, \delta)} \]

In the post-crisis market:
\[ \mathcal{L}(t, t, \delta) = \mathcal{L}_{\text{OIS}}(t, t, \delta) \]

We denote by $(r_t^T)_{t \geq 0}$ the short rate associated to OIS zero-coupon bonds.
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- **OIS rate**: fair swap rate for an Overnight Indexed Swap (proxy of risk-free rate in market practice).
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From OIS rates we can compute

- the term structure of OIS zero-coupon bond prices: $T \mapsto B(t, T)$;
- simply compounded OIS forward rates

\[
L^{\text{OIS}}(t, t, \delta) := \frac{1}{\delta} \left( \frac{1}{B(t, t + \delta)} - 1 \right).
\]

In the post-crisis market: $L(t, t, \delta) \neq L^{\text{OIS}}(t, t, \delta)$

- We denote by $(r_t)_{t \geq 0}$ the short rate associated to OIS zero-coupon bonds.
Besides $r_t$, we take **spot multiplicative spreads** as the main modeling quantities:

$$S^\delta(t, \cdot) := \frac{1 + \delta L(t, \cdot, \delta)}{1 + \delta L^{OIS}(t, \cdot, \delta)}, \quad \text{for } \delta \in \mathcal{D}.$$ 

- Directly **observable** from rates quoted on the market;
- **Market expectation at date** $t$ **of the interbank risk over** $[t, t + \delta]$;
- **Typical market behavior:**
  - $S^\delta_i(t, t) \geq 1$, for all $i = 1, \ldots, m$;
  - $S^\delta_i(t, t) \leq S^\delta_j(t, t)$, for all $i, j = 1, \ldots, m$ such that $\delta_i < \delta_j$. 

Let also define **forward multiplicative spreads**:

$$S^\delta(t, T) := \frac{1 + \delta L(t, T, \delta)}{1 + \delta L^{OIS}(t, T, \delta)}, \quad \text{for } 2D \text{ and } 0 \leq t \leq T,$$

where $L(t, T, \delta)$ is the **forward Ibor rate** (fair rate of a FRA).
Multiplicative spreads

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where \( L(t, T, \delta) \) is the forward Libor rate (fair rate of a FRA).

\( \Rightarrow \) compare with today’s talk by Ernst Eberlein.
A flow of CBI processes

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})\) be a filtered probability space supporting:

- a white noise \(W(ds, du)\) on \((0, +\infty)^2\) with intensity \(ds \, du\);
- a Poisson time-space random measure \(M(ds, dz, du)\) on \((0, +\infty)^3\) with intensity \(ds \pi(dz) \, du\) and compensator \(\tilde{M}(ds, dz, du)\).

For each \(i = 1, \ldots, m\), let \(Y^i = (Y^i_t)_{t \geq 0}\) be the unique strong solution of

\[
Y^i_t = y^i_0 + \int_0^t (\beta(i) - bY^i_s)ds + \sigma \int_0^t \int_0^t Y^i_s W(ds, du) \\
+ \eta \int_0^t \int_0^\infty \int_0^t Y^i_s \tilde{M}(ds, dz, du),
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where \(\{Y^i; i = 1, \ldots, m\}\) is a flow of CBI processes (see Dawson & Li, 2012).

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+ \eta \int_0^t \int_0^{+\infty} \int_0^\infty Y^i_s^- z \tilde{M}(ds, dz, du),
\]

where
- \(\beta: \{1, \ldots, m\} \to \mathbb{R}_+,\) with \(\beta(i) \leq \beta(i + 1)\);
- \((b, \sigma, \eta) \in \mathbb{R}^3_+;\)
- \(\pi\) is a tempered alpha-stable measure, explicitly given by

\[
\pi(dz) = \frac{1}{\Gamma(-\alpha) \cos(\pi \alpha/2)} \frac{e^{-\theta z}}{z^{1+\alpha}} 1\{z > 0\} dz,
\]

for some parameters \(\alpha \in (1, 2)\) and \(\theta > \eta\).

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Modeling multiple curves via a flow of CBI processes

Given the flow of CBI processes \( Y = \{ Y^i; i = 1, \ldots, m \} \), we specify the OIS short rate and spot multiplicative spreads as

\[
    r_t = \ell(t) + \mu^T Y_t, \\
    \log S^\delta_i(t, t) = c_i(t) + Y^i_t,
\]

for all \( t \geq 0 \) and \( i = 1, \ldots, m \), where \( \ell : \mathbb{R}_+ \to \mathbb{R} \) and \( c_i : \mathbb{R}_+ \to \mathbb{R}_+ \).
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- The functions \( \ell \) and \( c_i \) are chosen to fit the term structures at \( t = 0 \);
- spreads are by construction greater than one;
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- spreads are by construction greater than one;
- each process \( Y^i \) is a self-exciting mean-reverting process;
- the processes \( \{ Y^i; 1, \ldots, m \} \) are driven by the same sources of randomness;
- strong dependence among different spreads and OIS rates;
- spreads have a mutually exciting behavior: a large value of \( S^{\delta_i}(t, t) \) increases the likelihood of upward jumps of all spreads with tenor \( \delta_j > \delta_i \).
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$$r_t = \ell(t) + \mu^\top Y_t,$$

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**Proposition (monotonicity of spreads)**

Suppose that $c_i(t) \leq c_{i+1}(t)$ and $y_0^i \leq y_0^{i+1}$, for all $i = 1, \ldots, m - 1$ and $t \geq 0$. Then $S^{\delta_i}(t, T) \leq S^{\delta_{i+1}}(t, T)$ a.s., for all $i = 1, \ldots, m - 1$ and $0 \leq t \leq T < +\infty$. 
A sample path: OIS rate

Compare also with Jiao et al. (2017), sovereign interest rate modeling.
A sample path: multiplicative spreads
The affine property

CBI processes belong to the class of **affine processes** (Duffie et al., 2003):

\[
\mathbb{E}[e^{-pY_t^i}] = \exp \left( -y_0^i \nu(t, p) - \beta(i) \int_0^t \nu(s, p) \, ds \right), \quad \text{for all } t \geq 0,
\]

where the function \( \nu(\cdot, p) \) is given by the unique solution to the ODE

\[
\partial_t \nu(t, p) = -\phi(\nu(t, p)), \quad \nu(0, p) = p,
\]

with

\[
\phi(z) = bz + \frac{\sigma^2}{2} z^2 + \theta\alpha + z\alpha\eta\theta^{-1} - (z\eta + \theta)^\alpha, \quad \text{for } z \geq -\theta/\eta.
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\[ \phi(z) = b z + \frac{\sigma^2}{2} z^2 + \frac{\theta^\alpha + z \alpha \eta \theta^{\alpha-1} - (z \eta + \theta)^\alpha}{\cos(\pi \alpha/2)}, \quad \text{for } z \geq -\theta/\eta. \]

By relying on the affine property, we study the following features of the model:

- **existence of exponential moments** of \( Y_i \). In particular,

\[ b \geq \frac{\sigma^2}{2} \frac{\theta}{\eta} + \eta \frac{(1 - \alpha) \theta^{\alpha-1}}{\cos(\pi \alpha/2)} \quad \Rightarrow \quad \mathbb{E}[e^{Y^i_T}] < +\infty \quad \text{for all } T \geq 0. \]

- **0 is an inaccessible boundary** for \( Y_i \) if and only if \( \beta(i) \geq \sigma^2/2; \)

- **characterization of the ergodic distribution** of the flow.
OIS bond prices and forward multiplicative spreads

The affine property is crucial for **pricing applications**:  

1. **OIS zero-coupon bond** prices are given by

   \[ B(t, T) = \exp \left( A_0(t, T) + B_0(T - t)\top Y_t \right) \]

2. **forward multiplicative spreads** are given by

   \[ S^{\delta_i}(t, T) = \exp \left( A_i(t, T) + B_i(T - t)\top Y_t \right), \]

for all \( i = 1, \ldots, m \) and \( 0 \leq t \leq T < +\infty \).
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   \[ S_{\delta i}(t, T) = \exp\left( A_i(t, T) + B_i(T - t)^T Y_t \right), \]

for all \( i = 1, \ldots, m \) and \( 0 \leq t \leq T < +\infty \).

These formulae allow for a direct evaluation of linear interest rate derivatives:

- **Forward Rate Agreements**:

  \[ \Pi_{\text{FRA}}^\delta(t; T, \delta_i, K, N) = N(B(t, T)S_{\delta i}(t, T) - (1 + \delta_i K)B(t, T + \delta_i)); \]

- **Interest Rate Swaps and Basis Swaps**;

- **Convexity adjustments** of the form \( \mathbb{E}[L(T, T, \delta_i)|\mathcal{F}_t] - L(t, T, \delta_i) \).
Caplet pricing

Non-linear interest rate derivatives can be efficiently priced by combining
- knowledge of the characteristic function of the CBI flow;
- Fourier inversion techniques.

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Consider a Caplet with payoff \((L(T, T, \delta_i) - K)^+\) delivered at time \(T + \delta_i\):

\[
\Pi_{\text{CPL}}(t; T, \delta_i, K) = B(t, T + \delta_i) \mathbb{E}^{T + \delta_i} \left[ (e^{\mathcal{X}_T^i} - \bar{K}_i)^+ \bigg| \mathcal{F}_t \right],
\]

where \(\mathcal{X}_T^i := \log(S^\delta_i(T, T)/B(T, T + \delta_i))\) and \(\bar{K}_i := 1 + \delta_i K\).
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Consider a **Caplet** with payoff \((L(T, T, \delta_i) - K)^+\) delivered at time \(T + \delta_i\):

\[
P^{\text{CPL}}(t; T, \delta_i, K) = B(t, T + \delta_i) \mathbb{E}^{T+\delta_i} \left[ (e^{X^i_T} - \bar{K}_i)^+ | \mathcal{F}_t \right],
\]

where \(X^i_T := \log(S^{\delta_i}(T, T)/B(T, T + \delta_i))\) and \(\bar{K}_i := 1 + \delta_i K\). Let

\[
\Phi^i_{t, T}(\zeta) := B(t, T + \delta_i) \mathbb{E}^{T+\delta_i} [e^{i\zeta X^i_T} | \mathcal{F}_t]
\]

be the modified characteristic function of \(X^i_T\), which can be explicitly computed. Then

\[
P^{\text{CPL}}(t; T, \delta_i, K) = R^i_{t, T}(\bar{K}_i) + \frac{1}{\pi} \int_{0-i\epsilon}^{\infty-i\epsilon} \text{Re} \left( e^{-i\zeta \log(\bar{K}_i)} \frac{\Phi^i_{t, T}(\zeta - 1)}{-\zeta(\zeta - 1)} \right) d\zeta,
\]

where \(R^i_{t, T}(\bar{K}_i)\) is a (possibly null) residue term depending on \(\epsilon\).

Compare also with Lee (2004), Cuchiero et al. (2019).
Conclusions and outlook

- CBI processes allow to reproduce most of the empirical features of multi-curve spreads in post-crisis interest rate markets:
  - volatility clustering;
  - strong comovements of spreads;
  - persistence of low/negative rates.
- the affine property leads to efficient valuation techniques;
- Work in progress: calibration to market data on caps/floors volatility surface, with two tenors (OIS, 3M and 6M) by FFT and quantization methods.
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Thank you for your attention!